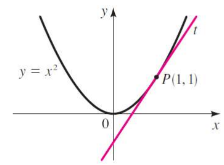
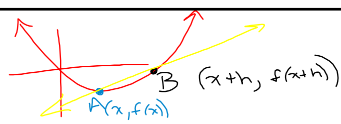
12.3 Tangent to a Curve

- A line *tangent* to a curve is the line that passes through the point and that has a slope that is the same as the slope of the curve at that point.

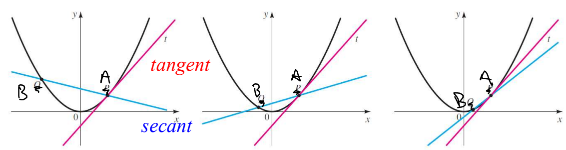




- Let A and B be two points near each other on the curve and let the coordinates of A be (x, f(x)). Suppose the x-coordinate of B differs from the x-coordinate of A by a small amount, h. Then the coordinates of B are (x + h, f(x + h))
- What is the slope of the line?

$$m=\frac{f(x+h)-f(x)}{x+h-x}=\frac{f(x+h)-f(x)}{h}, h\neq 0$$

- Suppose A remains fixed while B moves a long the curve toward A. Then the value of h will become smaller, approaching zero. Thus, h can be considered as a variable that approaches zero as B approaches A. If B is made to coincide with A, then the secant line becomes tangent to the curve at point A.



$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Find the slope of a line tangent to
y = 2x² – 3x + 1

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad m=4x-3$$

$$\lim_{h\to 0}\frac{2(x+h)^2-3(x+h)+1-(2x^2-3x+1)}{h}$$

$$\lim_{h\to 0}\frac{2(x^2+2xh+h^2)-3x-3h+1-2x^2+3x-1}{h}$$

$$\lim_{h\to 0}\frac{2x^2+4xh+2h^2-3h-2x^2}{h}=\lim_{h\to 0}\frac{4xh+2h^2-3h}{h}$$

$$=\lim_{h\to 0}(4x+2h-3)=4x-3$$

- The first derivative (f'(x)) of a function tells you the slope of all lines tangent to the function and is defined as:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Find the equation of the line tangent to the graph at the indicated point:
y = x^2 + x – 1 at (-4, 11)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{(x+h)^2+(x+h)-1-(x^2+x-1)}{h}$$

$$\lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-1-x^2-x+1}{h}$$

$$\lim_{h\to 0}\frac{2xh+h^2+h}{h}=\lim_{h\to 0}(2x+h+1)=2x+1$$

$$f'(x)=2(-4)+1$$
$$f'(x)=-7$$

$$y=mx+b$$
$$11=-7(-4)+b$$
$$11=28+b$$
$$-17=b$$
$$y=-7x-17$$

Find the equation of the line tangent to the graph at the indicated point:

$$y=\frac{4}{x}\text{ at }(4,\ 1)$$

$$f'(4)=-\frac{4}{(4)^2}=-\frac{4}{16}=-\frac{1}{4}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{\frac{4}{x+h}-\frac{4}{x}}{h}=\lim_{h\to 0}\frac{\frac{4x-4x-4h}{x(x+h)}}{h}$$

$$\lim_{h\to 0}\frac{-4h}{x(x+h)}\cdot\frac{1}{h}$$

$$\lim_{h\to 0}\frac{-4}{x(x+h)}=-\frac{4}{x^2}$$

$$y=mx+b$$
$$1=-\frac{1}{4}(4)+b$$
$$1=-1+b$$
$$2=b$$
$$y=-\frac{1}{4}x+2$$

Find the derivative of the function
f(x) = 8x^2 + 2x – 4 at x = 2.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{8(x+h)^2+2(x+h)-4-(8x^2+2x-4)}{h}$$

$$\lim_{h\to 0}\frac{8x^2+16xh+8h^2+2x+2h-4-8x^2-2x+4}{h}$$

$$\lim_{h\to 0}\frac{16xh+8h^2+2h}{h}$$

$$\lim_{h\to 0}(16x+8h+2)=(16x+2)$$

$$f'(2)=16(2)+2$$
$$f'(2)=34$$

12.3 b
Derivatives

Who uses this?
Quantitative Analyst
Financial Engineer
Fund Accountant
Engineering Researcher

- The derivative of a function f(x) is another function f'(x), that gives the **slope of the tangent line** to the function at any point.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The process of finding the derivative is called differentiation.
(this is one of the main parts of Calculus)

Another common notation for

$f'(x)$ is $\frac{dy}{dx}, y'$

This notation emphasizes that the derivative is a limit of slope, which is a change in y divided by a change in x.

Ex 1

Find an expression for the slope of the tangent line to the graph of y = $2x^2 - 3x + 4$ at any point.

$m = y' = 4x - 3$

- Find the slopes of the tangent lines when x = -1 and x = 5.

$y'(-1) = 4(-1) - 3 = -7$ $y'(5) = 4(5) - 3 = 17$

Rules:

1. **Constant:**

   f(x) = constant

   f′(x) = 0

2. ***Power***: if f(x) = x^n (n is a rational #)

   then f′(x) = nx^{n-1}

EX 2 f(x) = x^3

$f'(x) = 3x^{3-1}$

$f'(x) = 3x^2$

3. **Product of Constant & Power**:

   f(x) = cx^n

   then f′(x) = cnx^{n-1}

EX 3: f(x) = $4x^2$

$f'(x) = 8x$

4. **Sum & Difference:**

   if f(x) = g(x) +/- h(x)

   then f′(x)=g′(x) +/- h′(x)

EX 4: f(x) = $x^3 + 5x^2 + 6$

$f'(x) = 3x^2 + 10x + 0 = 3x^2 + 10x$

5. ***Product***:

$\frac{d(uv)}{dx}$ means the derivative of u times v both which are written in terms of x.

$$\frac{d(uv)}{dx} = \frac{v \bullet du}{dx} + \frac{u \bullet dv}{dx}$$

$$\frac{duv}{dx} = u' \cdot v + u \cdot v'$$

What does this mean?
Derivative of the first factor times the second factor plus the first factor times the derivative of the second factor.

Ex 5

$u = x^2 + 3$ $u' = 2x$

$v = 2x - 7$ $v' = 2$

- f(x) = $(x^2 + 3)(2x - 7)$

$f'(x) = (2x)(2x - 7) + (x^2 + 3)(2)$

$f'(x) = 4x^2 - 14x + 2x^2 + 6$

$f'(x) = 6x^2 - 14x + 6$

$u = x + 1$ $u' = 1$

$v = x^2 - 2x$ $v' = 2x - 2$

f(x)=$(x + 1)(x^2 - 2x)$

$f'(x) = (1)(x^2 - 2x) + (x + 1)(2x - 2)$

$f'(x) = x^2 - 2x + 2x^2 + 2x - 2x - 2$

$f'(x) = 3x^2 - 2x - 2$

6. ***Quotient***:

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\frac{v \bullet du}{dx} - \frac{u \bullet dv}{dx}}{v^2}$$

$$d\frac{u}{v} = \frac{u' \cdot v - u \cdot v'}{v^2}$$

u = x

v = x - 1

$$EX6: f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(1)(x-1) - (x)(1)}{(x-1)^2} = \frac{x - 1 - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$EX7: f(x)=\frac{2x^2+3}{5x^3-x}$$

$$f'(x)=\frac{(4x)\left(5x^3-x\right)-\left(2x^2+3\right)\left(15x^2-1\right)}{\left(5x^3-x\right)^2}$$

$u=2x^2+3$ $\quad$ $v=5x^3-x$

$u'=4x$ $\quad$ $v'=15x^2-1$

$$EX8: f(x)=\frac{\left(7x^2-1\right)x^3}{3x^2+2x}$$

$$f'(x)=\frac{\left(35x^4-3x^2\right)\left(3x^2+2x\right)-\left(7x^2-1\right)x^3\left(6x+2\right)}{\left(3x^2+2x\right)^2}$$

$u=\left(7x^2-1\right)x^3$ $\quad$ $v=3x^2+2x$

$u'=14x\left(x^3\right)+\left(7x^2-1\right)\left(3x^2\right)$ $\quad$ $v'=6x+2$

$u'=35x^4-3x^2$

7. Power of a Polynomial (*The Chain Rule*):

$$\frac{d\left(u^n\right)}{dx}=nu^{n-1}\frac{du}{dx}$$

$f(x)=(u)^n$

$f'(x)=n(u)^{n-1}\cdot u'$

EX 9: $f(x)=(3x-4)^3$

$f'(x)=3(3x-4)^2(3)$

$f'(x)=9(3x-4)^2$

Ex 10 $\quad f(x)=\left(3x^3-7x\right)\left(2x^4+5\right)^2$

$u=3x^3-7x$ $\quad$ $v=\left(2x^4+5\right)^2$

$u'=9x^2-7$ $\quad$ $v'=2\left(2x^4+5\right)^1 8x^3$

$$f'(x)=\left(9x^2-7\right)\left(2x^4+5\right)^2+\left(3x^3-7x\right)16x^3\left(2x^4+5\right)$$

Ex 11 $\quad f(x)=\sqrt{x^2-1}$

$f(x)=\left(x^2-1\right)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}(2x)$

$f'(x)=\frac{x}{\left(x^2-1\right)^{\frac{1}{2}}}$

$f'(x)=\frac{x}{\sqrt{x^2-1}}$

Ex 12 $\quad f(x)=\frac{x^2}{\left(x^2+1\right)^3}$ $\quad$ $u=x^2$ $\quad$ $u'=2x$

$v=\left(x^2+1\right)^3$ $\quad$ $v'=3\left(x^2+1\right)^2 2x$

$$f'(x)=\frac{2x\left(x^2+1\right)^3-x^2 6x\left(x^2+1\right)^2}{\left(x^2+1\right)^6}$$

$f(x)=\left(x^2\right)\left(x^2+1\right)^{-3}$

$f'(x)=(2x)\left(x^2+1\right)^{-3}+\left(x^2\right)\left[-3\left(x^2+1\right)^{-4}(2x)\right]$

$f'(x)=\frac{2x}{\left(x^2+1\right)^3}+\frac{-6x^3}{\left(x^2+1\right)^4}$

WS 12.3